On k-fold Intersections of Convex Sets

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Introduction

Reay's relaxed Tverberg conjecture is an open problem about the combinatorics of intersecting convex hulls. The problem asks for the minimum number of points in $\mathbb{R}^d$ that guarantees any such point set admits a partition into r parts, any k of whose convex hulls intersect. We give some new and improved lower bounds for this number, which Reay conjectured to be independent of k. We also prove a colored version of the conjecture for k sufficiently large.

Background

Given a finite point set in $\mathbb{R}^d$, the intersection pattern of convex hulls determined by subsets of those points is the focus of *Tverberg-type theory*. The namesake of the area, Helge Tverberg, established in 1966 that any $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ admit a partition into r parts $X_1, \ldots, X_r$ such that $\mathrm{conv}(X_1) \cap \ldots \cap \mathrm{conv}(X_r) \neq \emptyset$, and that this number of points is optimal in general [5].

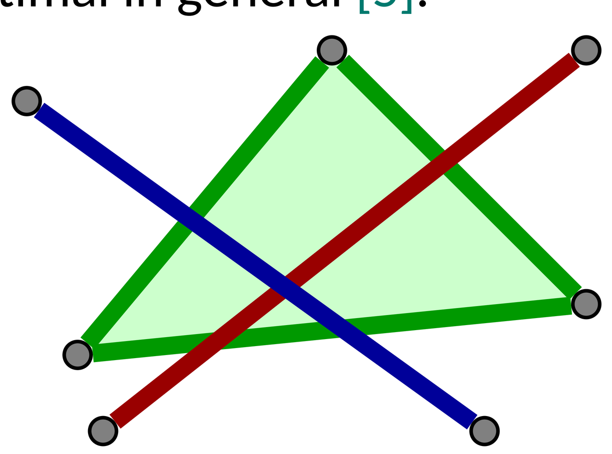
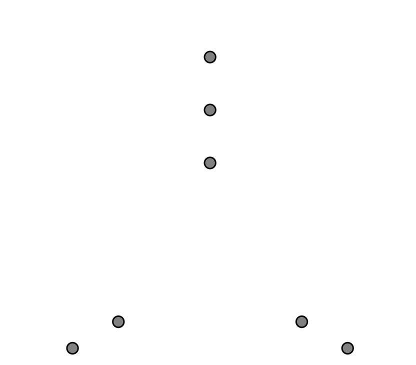


Figure: A Tverberg partition of seven points in $\mathbb{R}^2$ into three sets with intersecting convex hulls.

There are many open problems of interest concerning extensions or variations on Tverberg's original theorem:

- The Bárány-Larman conjecture asks whether a Tverberg partition exists if $r(d+1)$ points are assigned $d+1$ arbitrary color classes of size r and require the partitioned sets X_i to be "rainbow", i.e. contain at most one point of each color.
  - The non-trivial cases of the problem are known to be true for $r+1$ a prime, but it is open otherwise [1].
- Reay's problem asks whether the number of points required can be reduced if we only require k-fold intersections — Reay conjectured that the number of points cannot be reduced.
  - Some special cases are resolved. Examples in low dimension can be obtained by selecting points on simplicial axes [2]:



A Lower Bound from Strong General Position

Definition: Let $T(d, r, k)$ be the minimum number of points in $\mathbb{R}^d$ that guarantees any such point set admits a partition into r parts, any k of whose convex hulls intersect, as in Reay's Problem.

Definition: A point set X in general position is said to be in *strong general position* if for any disjoint subsets $X_1, \ldots, X_r \subset X$ with non-empty intersection the codimension of $\cap_i \mathrm{aff}(X_i)$ is equal to the sum of the codimensions of $\mathrm{aff}(X_i)$. See for example Perles and Sigron [3].

Theorem 1: $T(d, r, k) \geq r(\frac{k-1}{k} \cdot d + 1)$.

This result is obtained by counting dimensions and considering points in strong general position.

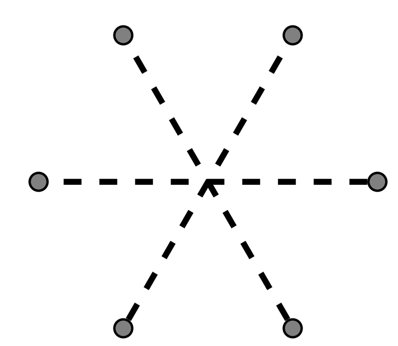


Figure: General position

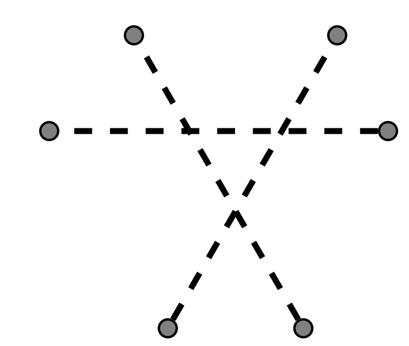
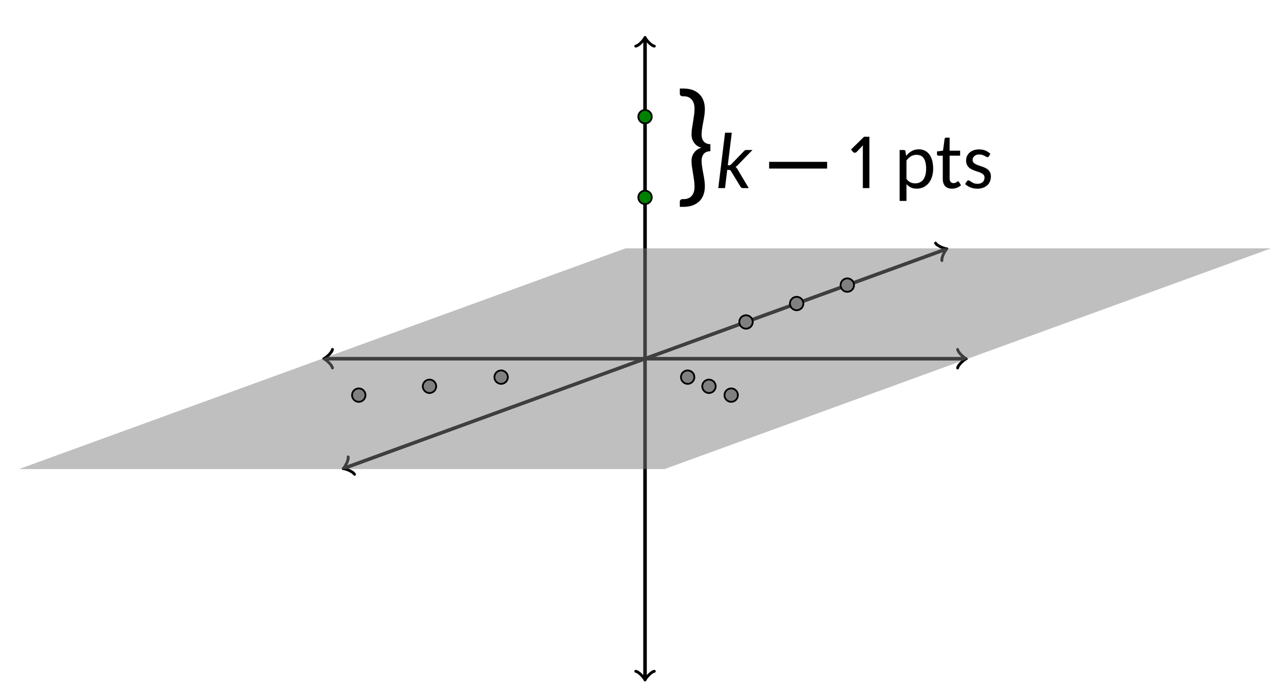


Figure: Strong general position

A Recursive Lower Bound

Theorem 2: $T(d, r, k) \geq T(d-1, r, k) + k - 1$.

Similar to the use of simplicial axes, we add $k-1$ points to an embedding of any example configuration in $\mathbb{R}^{d-1}$ into $\mathbb{R}^d$ to obtain an example configuration in $\mathbb{R}^d$.



Colorful Reay's Conjecture

We state a colorful analog of Reay's Conjecture, in the spirit of the Bárány-Larman conjecture, and prove a special case:

Conjecture 3.1: Given d, r, k there are point sets $C_1, \ldots, C_d \subset \mathbb{R}^d$ of cardinality r and C_0 of cardinality $r-1$ such that for any r pairwise disjoint sets $X_1, \ldots, X_r \subset \bigcup C_i$ with $|X_i \cap C_j| \leq 1$ for every i, j, the convex hulls of some k of them have empty intersection.

Theorem 3.2: *Conjecture 3.1* is true for $k > \lceil \frac{r}{2} \rceil$.

The proof of this uses similar ideas as in the recursive lower bound for $T(d, r, k)$, by embedding example configurations in $\mathbb{R}^{d-1}$ into $\mathbb{R}^d$ and adding additional points above and below the embedded hyperplane.

Future Research

- One proof of Tverberg's original theorem uses the *colorful Carathéodory theorem*, another statement about r-fold intersections of convex sets[4]. An appropriate pairwise analog of the theorem should make progress on Reay's conjecture, but we were unable to prove any sufficiently generalizable analogs of the theorem.
- One can also reformulate Reay's problem in terms of partitions of points in space by linear hyperplanes via the Gale dual, which may provide alternate methods of approach.

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